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Self-induced transparency with level degeneracy

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Abstract. The propagation of ultrashort coherent light pulses through a resonant atomic medium, modelled by a two-quantum level of absorbers, is studied including the effects of level degeneracy. The governing equations, a sequence of n-tuple sine-Gordon (SG) equations, are solved by quadrature for the Q-transition, with an approximate method indicated for the (mathematically) more difficult P- and R-transitions. It is shown, in particular, that there is a surprisingly close relation between the usual SG equation and the double SG equation. A simple theorem is proved which shows that each travelling wave solution of the former immediately leads to a solution of the latter, which thus admits a restricted Bäcklund transformation. This direct solution of nonlinear dispersive equations, which apparently cannot be solved by the inverse method, has an interesting by-product: an approximate method for finding travelling wave solutions of relatively arbitrary Klein-Gordon equations.

1. Introduction

An interesting and physically important problem in nonlinear optics is the self-induced transparency phenomenon in which ultrashort (shorter than the relevant dissipative relaxation time) coherent light pulses can propagate through a resonant atomic medium as if the medium were transparent. A tractable model of this resonant interaction of intense light with matter is obtained by considering the interaction between a two-quantum level system of absorbers and a light wave, the magnitude of which is written as a rapidly oscillating travelling wave with a more slowly varying envelope, whose frequency is almost exactly equal to a transition frequency between two populated energy levels of the atoms of the resonant medium. The leading edge of the sufficiently intense pulse inverts the population while the trailing edge returns the population to its initial state by stimulated emission. The pulse envelope propagates at a velocity much less than the phase velocity of light in the medium. Detailed discussions of the model have been given by McCall and Hahn [1], Lamb [2, 3] and others.

The aforementioned model is governed essentially by the sine-Gordon (SG) equation, an extensively studied equation, for which there is a Bäcklund transformation (BT) and a nonlinear superposition principle (theorem of permutability) [4], which permits additional solutions to be generated from known solutions purely algebraically. Further, there is a classical single-soliton solution which corresponds to a 2π-pulse solution of the model. Lamb [3] and Barnard [5] have used the nonlinear superposition principle to generate $2N\pi$ pulses, pulses for which the total area under the pulse envelope is $2N\pi$, by iteration. These larger-area pulses correspond to repeated inversions of the resonant medium, and as the alternating amplification and attenuation occurs, the pulses will split up, specifically into N stable 2π pulses. The iterated solutions exhibit exactly this behaviour.

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The model suffers from the defect of ignoring possible level degeneracy, i.e. the existence of more than one wavefunction associated with one or both of the population levels. The modifications needed for a particular degenerate two-level system are discussed in Lamb [6]. In this modified model, the SG equation is replaced by an n-tuple SG equation, the solution of which for the Q-branch transition is the primary topic of this paper. In actuality, a somewhat more general equation is considered, and an interesting by-product of the analysis is an algorithm for finding an approximate travelling wave solution of an arbitrary Klein–Gordon (KG) equation.

A brief derivation of the equations governing the non-degenerate model is given together with the modifications needed for the particular degenerate model. After reviewing one method of solving the SG equation, which is crucial to the later work, the $Q(2)$-branch transition, governed by a double SG equation, is solved, and, in particular, it is shown that there is a surprisingly close relation between this equation and the usual SG equation. Specifically, each travelling wave solution of the latter immediately leads to a travelling wave solution of the former. Further, the double SG equation admits a BT, the zero solution of which leads to an alternative method of determining the single-soliton solution.

Explicit solutions are obtained for n-tuple SG equations for the cases $n=3$, 4, 5, and the case of general n is discussed. The solution may be obtained by finding the roots of an algebraic equation of order $n-1$, after which the solution follows by solving a system of linear algebraic equations and then performing a quadrature.

2. The basic equations

The equations governing the model without level degeneracy are

$$\tau_t + c_1\tau_x = \alpha_1^2 S \tag{1}$$

$$S_t = \tau T \tag{2}$$

$$T_t = -\tau S \tag{3}$$

where the functions S and T are evaluated at zero frequency shift, and the translational motion of the atoms has been eliminated. The optical field O and the macroscopic polarization density P are written as

$$O(x, t) = (\tau\hbar/p)\cos(\hat{k}x - \omega t) \qquad P(x, t) = N_0 pS\sin(\hat{k}x - \omega t)$$

where x is distance, t is the time, $T = N/N_0$, N is the population inversion, N_0 is the number of active atoms per unit volume, c_1 is the phase velocity of light in the medium, p is the dipole matrix element, $\alpha_1^2 = 2\pi N_0\omega p^2/\hbar$ and $\hbar$ is Plank's constant divided by 2π. Partial derivatives are denoted by subscripts. Equations (2) and (3) are satisfied by $S^2 + T^2 = 1$ so that

$$S = \pm\sin\sigma \qquad T = \pm\cos\sigma \qquad \tau = \sigma_t. \tag{4}$$

Equation (1) then may be written as

$$\sigma_{\xi\eta} = \pm\sin\sigma \tag{5}$$

where $\xi = x\alpha_1/t$, $\eta = \alpha_1(t - x/c_1)$. The plus sign corresponds to the situation in which the population is initially inverted (propagation in an amplifier) while the minus sign corresponds to the atomic population at the lower level (propagation in an attenuator).

Since it has been shown that soliton propagation in an amplifier is unstable, the model has been utilized primarily with the lower sign in equation (5).

The classical single-soliton solution of equation (5) is given by (lower sign)

$$\sigma = 4\tan^{-1}\exp(\lambda) \tag{6}$$

where $\lambda = a\alpha_1(t - x/v_1)$, a is the Bäcklund parameter and $v_1 = c_1/(1+a^{-2})$; this gives the field as a hyperbolic secant pulse.

The modifications necessary for the particular degenerate two-level system are discussed in Lamb [6]; the equation which replaces equation (5) is

$$\sigma_{\xi\eta} = [-\alpha_1^2/c_1(2r+1)^{1/2}] \sum_{m=-j}^{j} d_m \sin(d_m\sigma) \tag{7}$$

where r is a quantum number (for a given r, there are $2r+1$ levels) and $j = r+s$, where s is the spin quantum number. For the primary discussion of the present paper, the coefficients have the values

$$d_m = m/j. \tag{8}$$

Equation (7) will be considered for arbitrary coefficients, with complete solutions presented only for the case given by equation (8); the case $n=2$ is an exception with complete solutions presented for arbitrary coefficients, including the $Q(2)$-transition. Lamb [6] has presented a discussion of the $Q(2)$-transition, using a solution $\sigma = \sigma(t - x/v_1)$, attributed to Seeger; references to some numerical studies are given.

3. The double sine-Gordon equation

The solution of the double SG equation will be related to the solution of the usual SG equation. Thus, it is appropriate to examine the latter in some detail before determining the solution of the former. A trial solution of

$$\sigma_{\xi\eta} = \sin\sigma \tag{9}$$

will be sought in the form

$$\sigma = 2\tan^{-1} F(A\xi + B\eta) \tag{10}$$

where F is assumed to be a twice continuously differentiable function and A and B are arbitrary constants. This assumption leads to the ordinary differential equation

$$L[F] = 1/AB \tag{11}$$

where

$$L[F] = F''/F - 2F'^2/(1+F^2) \tag{12}$$

and the prime denotes differentiation with respect to the argument. The operator L recurs throughout the paper.

Equation (11) is satisfied by

$$F'^2 = (E + EF^2 - 1/AB)(1+F^2) \tag{13}$$

where E is an arbitrary constant; a quadrature yields F. In particular, a solution for $F(A\xi + \eta/A)$ is obtained from

$$F'^2 = (E - 1 + EF^2)(1+F^2) \tag{14}$$

while a solution for $F(A\xi-\eta/A)$ is obtained from

$$F'^2=(E+1+EF^2)(1+F^2). \tag{15}$$

Note that the single-soliton solution of equation (9) is obtained by letting $E=1$ in equation (14). For later reference, the choice $E=0$ in equation (13) gives

$$F'^2=-(1+F^2)/AB. \tag{16}$$

The double SG equation will be taken in the form

$$\sigma_{\xi\eta}=-D^2(\alpha\sin(\sigma/2)+\beta\sin\sigma) \tag{17}$$

where (α,β) and D are constants. Before discussing the general case, some solutions for the $Q(2)$-transition will be considered, i.e. $(\alpha,\beta)=(\frac{1}{2},1)$.

A particular solution of equation (17) may be found by letting

$$w=\int \mathrm{d}\sigma/D[\cos\sigma+\cos(\sigma/2)]^{1/2}. \tag{18}$$

This gives a solution

$$w=A\xi+2\eta/A\equiv 2Y$$

which, since the integral in equation (18) may be evaluated directly, gives

$$\cos(\sigma/2)=[5-\sin(D(6)^{1/2}Y)][7+\sin(D(6)^{1/2}Y)]^{-1}$$

where A is an arbitrary constant. More detailed results may be obtained by the use of this transformation [7, 8].

Table 1 lists additional solutions of equation (17) in the form

$$\sigma=-4\tan^{-1}[cF(A\xi+B\eta)] \tag{19}$$

where F is a combination of Jacobian elliptic functions, with constants A and B, arbitrary modulus k and a yet to be determined constant c. The arguments of the functions will be omitted in order to simplify the notation. Equations (20)-(23) give four sets of periodic solutions, the first two of which have limiting solutions which are the single-soliton solution.

It is possible to solve equation (17) directly by looking for a solution in the form

$$\sigma=4\tan^{-1}[F(A\xi+B\eta)] \tag{24}$$

Table 1. Solutions of equation (17).

Trial solution F	Conditions	Limiting solution	Comments
1. $c\,cn/sn$	$3(k^2-1)c^4+(k^2-2)c^2+5=0$ $AB=-D^2c^2/4[(k^2-1)c^4+1]$	$-4\tan^{-1}[\sqrt{5}\,\mathrm{cosech}(A^*\xi+B^*\eta)]$ $A^*B^*=-\frac{5}{4}D^2$	Soliton limit (20)
2. $c\,dn/sn$	$3k^2(1-k^2)c^4+(1-2k^2)c^2+5=0$ $AB=-D^2c^2/4[1+k^2(1-k^2)c^4]$	Same as 1	Soliton limit (21)
3. $c\,sn/cn$	$3c^4+(2-k^2)c^2+5(k^2-1)=0$ $AB=-D^2c^2/4[c^4+k^2-1]$	None	Oscillatory solution (22)
4. $c\,sn/dn$	$3c^4+(2k^2-1)c^2-5k^2(k^2-1)=0$ $AB=-D^2c^2/4[k^2(k^2-1)-c^4]$	None	Oscillatory solution (23)

where F is now a general function. Equation (17) will be satisfied provided that

$$F'^2=(1+E_1F^2)(1+E_2F^2) \tag{25}$$

where

$$\begin{aligned} 4E_1&=4+\alpha+2\beta+[(\alpha+2\beta)^2+16\beta]^{1/2} \\ E_1E_2&=1+\alpha/2 \qquad AB=-D^2. \end{aligned} \tag{26}$$

In particular, when $(\alpha,\beta)=(\frac{1}{2},1)$,

$$8E_1=13+(89)^{1/2} \qquad 4E_1E_2=5. \tag{27}$$

The preceding discussion may be generalized by the following theorem.

Theorem. Let $\sigma=2\tan^{-1}F(A\xi+B\eta)$, with constant A and B, be a solution of the SG equation, equation (9). Then

$$\sigma=-4\tan^{-1}[cF(A_1\xi+B_1\eta)] \tag{28}$$

is a solution of the double SG equation, equation (17).

The values of A_1, B_1 and c are given in the body of the proof.

Proof. The proof is simple and constructive. Differentiation of equation (28) gives

$$\sigma_{\xi\eta}=2A_1B_1\sin(\sigma/2)[F''/F-2c^2F'^2/(1+c^2F^2)]. \tag{29}$$

Substitution of equation (13) into equation (29) gives

$$\sigma_{\xi\eta}=2A_1B_1\sin(\sigma/2)\{[2(1-c^2)F'^2/(1+F^2)(1+c^2F^2)]+1/AB\}$$

which will satisfy equation (17) provided that

$$\begin{aligned} &(-E+1/AB)c^4+(2E-1/AB)c^2-E=2\beta[(-E+1/AB)c^4+E]/\alpha \\ &A_1B_1=-D^2\alpha c^2/2[E+(-E+1/AB)c^4]. \end{aligned}$$

Noting that the single-soliton solution of the $Q(2)$-transition is obtained by choosing $E=1$, $AB=1$ gives $c^2=5$, $A_1B_1=-5D^2/4$, in agreement with the previous results; thus, the single soliton is obtained from

$$F'^2=5F^2(1+F^2).$$

Finally, the double SG equation, equation (17), admits a solution

$$\sigma=4\tan^{-1}F(A\xi+B\eta) \tag{30}$$

with $F'^2=1+EF^2$, provided that $E=(\alpha-2\beta)/\alpha$ and $AB=-D^2(\alpha+2\beta)/2(E-2)$.

4. The triple sine-Gordon equation

The equation will be taken in the form

$$\sigma_{\xi\eta}=-D^2[\alpha\sin(\sigma/3)+\beta\sin(2\sigma/3)+\gamma\sin\sigma]. \tag{31}$$

The analysis of the previous section provides the clue as to how to obtain a solution of equation (31). Thus, a trial solution will be taken in the form

$$\sigma = 6 \tan^{-1} F(A\xi + B\eta). \tag{32}$$

Substitution of equation (32) into equation (31) gives

$$\begin{aligned} L[F] &= D^2[\alpha + 3\gamma + 2\beta \cos(\sigma/3) - 4\gamma \sin^2(\sigma/3)]/3AB \\ &= D^2 T_3[1 + 2(\alpha - 5\gamma)F^2 T_3^{-1} + (\alpha - 2\beta + 3\gamma)F^4 T_3^{-1}]/3AB(1+F^2)^2 \end{aligned} \tag{33}$$

where $T_3 = \alpha + 2\beta + 3\gamma$. The function

$$(1+F^2)F'^2 = (1+E_1F^2)(1+E_2F^2) \tag{34}$$

will satisfy equation (33) provided that

$$E_1E_2T_3 - 2(E_1+E_2)(\alpha + \beta - \gamma) = -3\alpha + 15\gamma$$

$$E_1E_2T_3 + (E_1+E_2)(\alpha - 2\beta + 3\gamma) = 3(\alpha - 2\beta + 3\gamma).$$

This gives

$$(3\alpha + \gamma)E_i = 3(\alpha - \beta - \gamma) + (-1)^i[3(3\beta^2 + 8\beta\gamma - 16\alpha\gamma)]^{1/2}$$

$i = 1, 2$, which, together with the choice

$$AB = T_3D^2/3(E_1 + E_2 - 3)$$

reduces the solution to a quadrature of equation (34). For $(\alpha, \beta, \gamma) = (\frac{1}{3}, \frac{2}{3}, 1)$, $E_1 = -3$, $E_2 = -1$, $AB = -2D^2/9$.

It is possible to relate the solutions of the double and triple SG equations in the following way.

Let

$$\sigma = -4 \tan^{-1} F(A\xi + B\eta)$$

be a solution of

$$\sigma_{\xi\eta} = -D^2[U \sin(\sigma/2) - V \sin\sigma] \tag{35}$$

with constant (U, V). Then

$$F'^2 = (1 + E_1^*F^2)(1 + E_2^*F^2) \tag{36}$$

with $AB = -D^2/4n$, with constant n, will satisfy equation (35) if

$$E_1^* = 1 + n(U - 2V) + [n^2(U-2V)^2 - 4nV]^{1/2} \qquad E_2^* = (1 + 2nU)/E_1^*.$$

A direct calculation shows that

$$\sigma = 6 \tan^{-1} F(A_1\xi + B_1\eta)$$

will be a solution of equation (31) of the form of equation (34) with

$$\alpha = \tfrac{2}{3} - nV/6 \qquad \beta = 2n(V-U)/3 \qquad \gamma = nV/2 \qquad A_1B_1 = -2D^2/9.$$

The solution obtained previously is obtained by choosing $n = 2$, $(U, V) = (\frac{1}{2}, 1)$.

5. The quadruple sine-Gordon equation

The equation will be taken in the form

$$\sigma_{\xi\eta} = -D^2[\alpha \sin(\sigma/4) + \beta \sin(2\sigma/4) + \gamma \sin(3\sigma/4) + \delta \sin\sigma]. \tag{37}$$

Proceeding as in the previous sections, substitution of

$$\sigma = 8\tan^{-1} F(A\xi + B\eta) \tag{38}$$

into equation (37) gives

$$4AB(1+F^2)^3 L[F]/D^2 T_4$$
$$= [1 + (3\alpha + 2\beta - 7\gamma - 28\delta)F^2 T_4^{-1} + (3\alpha - 2\beta - 7\gamma + 28\delta)F^4 T_4^{-1} + (\alpha - 2\beta + 3\gamma - 4\delta)F^6 T_4^{-1}] \tag{39}$$

where $T_4 = \alpha + 2\beta + 3\gamma + 4\delta$. Equation (39) will be satisfied by

$$F'^2 = \prod_{i=1}^{3} (1 + E_i F^2)/(1+F^2)^2 \tag{40}$$

provided that

$$(3\alpha + 2\beta - 7\gamma - 28\delta)T_4^{-1} = (2R_2 - 3R_1)/(R_3 - 4)$$
$$(3\alpha - 2\beta - 7\gamma + 28\delta)T_4^{-1} = (3R_3 - 2R_2)/(R_1 - 4)$$
$$(\alpha - 2\beta + 3\gamma - 4\delta)T_4^{-1} = R_3/(4 - R_1)$$

where $R_1 = E_1 + E_2 + E_3$, $R_2 = E_1E_2 + E_1E_3 + E_2E_3$, $R_3 = E_1E_2E_3$. For a numerical solution for $(\alpha, \beta, \gamma, \delta) = (\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1)$, the problem reduces to solving the cubic equation

$$y_3^3 + 22y_3^2 + 136y_3 + 160 = 0$$

where $y_3 = -11 - E_1$. The roots are obtained easily, so that

$$E_1 = -1 \qquad E_2 = -5 + 2(5)^{1/2} \qquad E_3 = -5 - 2(5)^{1/2}$$
$$AB = T_4 D^2/(E_1 + E_2 + E_3 - 4)4.$$

6. The quintuple sine-Gordon equation

The equation will be taken in the form

$$\sigma_{\xi\eta} = -D^2[\alpha \sin(\sigma/5) + \beta \sin(2\sigma/5) + \gamma \sin(3\sigma/5) + \delta \sin(4\sigma/5) + \varepsilon \sin \sigma]. \tag{41}$$

Substitution of

$$\sigma = 10\tan^{-1} F(A\xi + B\eta) \tag{42}$$

into equation (41) gives

$$5AB(1+F^2)^4 L[F]/D^2 T_5$$
$$= [1 + 4(\alpha + \beta - \gamma - 6\delta - 15\varepsilon)F^2 T_5^{-1} + 2(3\alpha - 7\gamma + 63\varepsilon)F^4 T_5^{-1} + 4(\alpha - \beta - \gamma - 6\delta - 15\varepsilon)F^6 T_5^{-1} + (\alpha - 2\beta + 3\gamma - 4\delta + 5\varepsilon)F^8 T_5^{-1}] \tag{43}$$

where $T_5 = \alpha + 2\beta + 3\gamma + 4\delta + 5\varepsilon$. Equation (43) will be satisfied by the function

$$F'^2 = \prod_{i=1}^{4} (1 + E_i F^2)/(1+F^2)^3$$

provided that

$$\alpha+\beta-\gamma-6\delta-15\varepsilon=T_5(S_2-2S_1)/2(S_1-5)$$

$$2(3\alpha-7\gamma+63\varepsilon)=3T_5(S_3-S_2)/(S_1-5)$$

$$2(\alpha-\beta-\gamma-6\delta-15\varepsilon)=(2S_4-S_3)T_5/(S_1-5)$$

$$\alpha-2\beta+3\gamma-4\delta+5\varepsilon=S_4T_5/(5-S_1)$$

where

$$S_1=E_1+E_2+E_3$$

$$S_2=E_1E_4+E_1E_2+E_2E_3+E_2E_4+E_3E_4+E_1E_3$$

$$S_3=E_1E_2E_3+E_1E_2E_4+E_1E_3E_4+E_2E_3E_4$$

$$S_4=E_1E_2E_3E_4.$$

For $(\alpha, \beta, \gamma, \delta, \varepsilon)=(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1)$, the problem is reduced to solving the quartic equation

$$y_4^4-\tfrac{40}{3}y_4^3-\tfrac{10}{3}y_4^2-\tfrac{80}{3}y_4-\tfrac{25}{11}=0 \qquad y_4=\tfrac{40}{3}-(E_1+E_2+E_3).$$

The roots are obtained easily and give

$$E_1=-0.085\,83 \qquad E_2=13.718\,876$$

$$E_3=-0.149\,8565+\mathrm{i}(1.381\,1959) \qquad E_4=-0.149\,8565-\mathrm{i}(1.381\,1959)$$

which, together with

$$AB=D^2T_5/(E_1+E_2+E_3+E_4-5)5$$

reduce the solution of the problem to a quadrature. This is the first case of a Q-transition in which all of the roots are not real.

7. The general case

The equation for an arbitrary number of terms may be written as

$$\sigma_{\xi\eta}=-D^2\sum_{i=1}^{n}a_i\sin(i\sigma/n). \tag{44}$$

The solution is taken in the form

$$\sigma=2n\tan^{-1}F(A\xi+B\eta) \tag{45}$$

which leads to

$$F'^2=\prod_{i=1}^{n-1}(1+E_iF^2)/(1+F^2)^{n-2} \tag{46}$$

where the E_i, $i=1, 2, \ldots, n-1$, are the roots of an $(n-1)$th-order algebraic equation, obtained as illustrated in the specific examples. A and B satisfy

$$AB=\sum_{i=1}^{n}(ia_i)D^2/n\sum_{k=1}^{n-1}(E_k-n). \tag{47}$$

Equations (46) and (47) are valid for $n>2$. Structurally, they are also valid for $n=2$ (equation (30)) and $n=1$ (equation (16)), but these two cases require the negative of AB as given by equation (47).

This solution may be used in another way. Suppose a rather general KG equation is considered, say

$$\sigma_{\xi\eta} = v'(\sigma). \tag{48}$$

Assuming that the function $v'(\sigma)$ may be expanded into a Fourier sine series

$$v'(\sigma) = \sum_{r=1}^{\infty} b_r \sin(r\sigma) \tag{49}$$

where $-\pi \leqslant \sigma \leqslant \pi$ has been chosen for convenience, the change of variable $\sigma = \theta/n$ transforms equation (48) to

$$\theta_{\xi\eta}/n = \sum_{r=1}^{\infty} b_r \sin(r\theta/n) \tag{50}$$

which is clearly of the form of the equations considered. Consequently, the method of the paper provides a way to obtain an approximate solution for a travelling wave for a rather general KG equation when the series in equation (50) is approximated by a partial sum of order n.

8. The Bäcklund transformation for the double sine-Gordon equation

The usual SG equation, equation (9), admits an auto-BT relating two solutions σ_1 and σ_2; specifically,

$$\begin{aligned} \frac{\partial}{\partial\xi}\left(\frac{\sigma_1-\sigma_2}{2}\right) &= a \sin\left(\frac{\sigma_1+\sigma_2}{2}\right) \\ \frac{\partial}{\partial\eta}\left(\frac{\sigma_1+\sigma_2}{2}\right) &= a^{-1} \sin\left(\frac{\sigma_1-\sigma_2}{2}\right). \end{aligned} \tag{51}$$

The close relation between the single and double SG equations, as exemplified by the theorem of section 3 means that a restricted BT may be written relating two solutions of the latter. There are various ways of writing a transformation valid for the double SG equation, but the most practical is to write equations (51) in terms of $\sigma = 2\tan^{-1} F(A\xi \pm \eta/A)$; this gives

$$\begin{aligned} F_{1\xi}/(1+F_1^2) - F_{2\xi}/(1+F_2^2) &= a(F_1+F_2)/[(1+F_1^2)(1+F_2^2)]^{1/2} \\ F_{1\eta}/(1+F_1^2) + F_{2\eta}/(1+F_2^2) &= (F_1-F_2)/a[(1+F_1^2)(1+F_2^2)]^{1/2}. \end{aligned} \tag{52}$$

Equations (52) provide a BT for the double SG equation if F is taken to be determined from

$$\tfrac{1}{4}\sigma = -\tan^{-1} cF[\lambda_1(A\xi \pm \eta/A)]$$

with constant λ_1. Letting $F_2 = 0$ leads to

$$F_1 = -\text{cosech}(a\xi \pm \eta/a).$$

Thus

$$\sigma = 4\tan^{-1}[c \text{ cosech } \lambda_1(a\xi \pm \eta/a)]$$

is the corresponding solution of the double SG equation. This gives an alternative method of determining single-soliton solutions of the double SG equation.

9. The *P*- and *R*-transitions

The coefficients in equation (7) have the values

$$d_m = (j^2 - m^2)^{1/2}/j \qquad d_m = [(j+1)^2 - m^2]^{1/2}/j \tag{53}$$

for these transitions.

These coefficients lead to mathematically more difficult equations since the arguments of the sine functions are no longer all rational. For example, for $j = 2$, the equations would be

$$\sigma_{\xi\eta} = -D^2[\sin\sigma + \sqrt{3}\sin(\sqrt{3}\sigma/2)] \tag{54}$$

$$\sigma_{\xi\eta} = -D^2[\tfrac{3}{2}\sin(3\sigma/2) + 2\sqrt{2}\sin(\sqrt{2}\sigma) + \sqrt{5}\sin(\sqrt{5}\sigma/2)] \tag{55}$$

The following procedure would give an approximate solution. Since

$$\sin b\sigma = -(2\pi^{-1}\sin b\pi)[\sin\sigma/(b^2-1) - 2\sin 2\sigma/(b^2-2^2) + 3\sin 3\sigma/(b^2-3^2) + \ldots] \tag{56}$$

for $-\pi \leqslant \sigma \leqslant \pi$, each of the sine terms may be expanded in sine series using equation (56). This would lead to equations of the form

$$\sigma_{\xi\eta} = -D^2[A_1\sin\sigma + A_2\sin(2\sigma) + \ldots] \tag{57}$$

with constant A_i. The change of variable $\sigma = \theta/n$ transforms equation (57) to

$$\theta_{\xi\eta} = -nD^2[A_1\sin(\theta/n) + A_2\sin(2\theta/n) + \ldots]. \tag{58}$$

If the series obtained from the use of equation (56) are approximated by partial sums of order n, equation (58) becomes an equation of the type considered in the paper.

As an example, for $n = 7$, the equation corresponding to equation (54) would be

$$\theta_{\xi\eta} = -7D^2[1.451\sin(\theta/7) - 0.277\sin(2\theta/7) + 0.164\sin(3\theta/7) - 0.118\sin(4\theta/7) + 0.0929\sin(5\theta/7) - 0.0769\sin(6\theta/7) + 0.0654\sin\theta].$$

10. Conclusions

Analytical solutions have been obtained for a class of n-tuple SG equations which govern the Q-transition. Explicit solutions are given up through the five-tuple equation together with a formula for the general case, the method of obtaining the requisite algebraic equation being clear from the explicit examples. Since the method is applicable to n-tuple SG equations with arbitrary coefficients, it may be utilized to give an approximate solution for the P- and R-transitions after the appropriate sine terms have been approximated by partial sums of their Fourier sine series.

The theorem connecting the solutions of the SG and the double SG equation shows that the latter admits a restricted BT, the 'vacuum' solution of which is the single-soliton solution. This result is more in harmony with the classical theory than the usual constant-phase method. Analogous results hold true for the triple SG equation.

References

[1] McCall S and Hahn E 1969 Self-induced transparency *Phys. Rev.* **183** 457-85

[2] Lamb G Jr 1971 Analytical descriptions of ultrashort optical pulse propagation in a resonant medium *Rev. Mod. Phys.* **43** 99-124

[3] Lamb G Jr 1967 Propagation of ultrashort optical pulses *Phys. Lett.* **25A** 181-2

[4] Rodgers C and Shadwick W 1982 *Bäcklund Transformations and their Applications* (New York: Academic)

[5] Barnard T 1973 $2N\pi$ ultrashort light pulses *Phys. Rev.* A 7 373-6

[6] Lamb G Jr 1980 *Elements of Soliton Theory* (New York: Wiley)

[7] Gundersen R 1990 Travelling wave solutions of nonlinear partial differential equations *J. Eng. Math.* to appear

[8] Gundersen R 1990 Solutions of generalized sine-Gordon equations *Preprint*